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The Signature of the Symmetric Union of Links

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§ 1. Introduction

Let L be an unoriented link in R^3 and $D(L)$ a link diagram of L on R^2. First we assume that $D(L)$ is connected. R^2 is divided into connected regions by $D(L)$ which can be alternately colored white and black in such a way that adjacent regions are of distinct colors and the unbounded region is a white region. For each crossing P of $D(L)$, if the white regions at P are distinct, we assign the incident number $\theta(\mathrm{P})$ with 1 or -1 depending on whether the overcrossing line coincides with the undercrossing line by turning over the black regions at P counterclockwise or clockwise. If the white regions at P are same, then we define $\theta(\mathrm{P})$ to be 0. Let $W_0, W_1, W_2, \cdots, Wm$ be the white regions. For $0 \leq i \leq m$ and $0 \leq j \leq m$, we define g_{ij} as follows:

$$g_{ij} = \begin{cases} \sum\limits_{\mathrm{P} \in \partial W_i} \theta(\mathrm{P}) & if\ i = j, \\ -\sum\limits_{\mathrm{P} \in W_i \cap W_j} \theta(\mathrm{P}) & if\ i \neq j. \end{cases}$$

Define the $(m+1) \times (m+1)$ matrix $\tilde{G}(L)$ by $\tilde{G}(L) = (g_{ij})$. The $m \times m$ matrix $G(L)$ obtained from $\tilde{G}(L)$ by deleting the row and the column corresponding to any white region W_{i_0} is called *the Goeritz matrix associated to* $D(L)$ (see [2], [4], [5] and [11]).

If $D(L)$ is disconnected, then $D(L)$ can be expressed as the union of a finite number of link diagrams $D(L_1), D(L_2), \cdots, D(L_n)$ such that $D(L_1), D(L_2), \cdots, D(L_n)$ are connected and are disjoint of each other. We construct $\tilde{G}(L_1), \tilde{G}(L_2), \cdots, \tilde{G}(L_n)$ for link diagrams $D(L_1), D(L_2), \cdots, D(L_n)$, respectively. Then we define the matrix $\tilde{G}(L)$ by

$$\tilde{G}(L) = \begin{pmatrix} \tilde{G}(L_1) & 0 & \cdots & 0 \\ 0 & \tilde{G}(L_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \tilde{G}(L_n) \end{pmatrix}.$$

The matrix $G(L)$ obtained from $\tilde{G}(L)$ by deleting the i_0-th row and the i_0-th column for any i_0 is called *the Goeritz matrix associated to* $D(L)$ for this case.

Now let L be an oriented link and $D(L)$ its diagram. A crossing P of $D(L)$ is said to be *of type* II if the oriented overcrossing line coincides with the oriented undercrossing line by turning over the black regions at P. Otherwise, it is said to be *of type* I. We define $\mu(D(L)) = \sum \theta(\mathrm{P})$ (summed over all crossings P of type II).

For a symmetric matrix A, $\sigma(A)$ will denote the signature of A, i.e., when A is diagonalized, the signature of A is defined to be the number of positive diagonal elements minus the number of negative diagonal elements (see [8], [10]). It is well-known that for an oriented link L, $\sigma(G(L)) - \mu(D(L))$ is an invariant of the oriented link type of L, which is denoted by $\sigma(L)$ and is called *the signature of an oriented link* L (see[4], [6]). In [4], Y. Furuta and Y. Shinohara showed that for an oriented link L and its diagram $D(L)$, $\chi(L) = \mathrm{rank}G(L) - \mu(D(L))$ (mod 2) is an invariant of the oriented link type of L.

Let L be a link in R^3. As shown in Fig. 1.1, first we take a connected sum of L and its mirror image L^*, and secondly we twist two corresponding parallel arcs, one from L and the other from L^*, n times. Here, the twisting number n is defined to be positive or negative according as shown in Fig. 1.2. The resulting link is denoted by $S(L, n)$ and is called *a symmetric union* if n is even and *a skew-symmetric union* if n is odd.

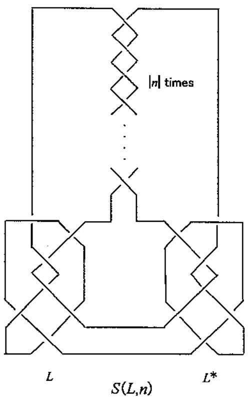

Fig. 1.1

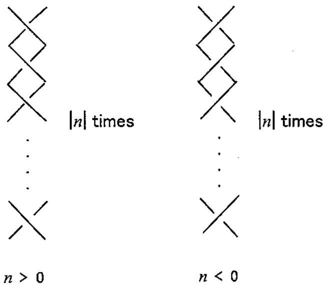

Fig. 1.2

The symmetric union and the skew-symmetric union were introduced by S. Kinoshita and H. Terasaka in [9]. For the case that L is a trivial knot, R. Fukumoto and Y. Shinohara computed the Lickorish-Millett polynomial for $S(L, 2n)$, by using which they classified $S(L, 2n)$ in [3]. In this paper, we will study the signature of $S(L, n)$ and obtain the following theorem:

THEOREM 1.1 *If $S(L, n)$ is a symmetric union of an oriented link L, i.e., n is even, then we have*

$$\sigma\left(S(L,n)\right) = \begin{cases} 0 & \textit{if} \quad \chi\left(S(L,n)\right) \equiv 0 \pmod 2, \\ -\epsilon & \textit{if} \quad \chi\left(S(L,n)\right) \equiv 1 \pmod 2, \end{cases}$$

where ϵ denotes the sign of n.

We also give an alternate proof for the following theorem which was proved by S. Asakura and Y. Shinohara [1]:

THEOREM 1.2 *If n is even, then for any oriented link L with* $\det L \neq 0$ *we have* $\sigma\left(S(L, n)\right) = 0$.

§ 2. The Signature of the Goeritz Matrix of $S(L, n)$

The link diagram $D(S(L, n))$ of $S(L, n)$ is given in Fig. 2.1.

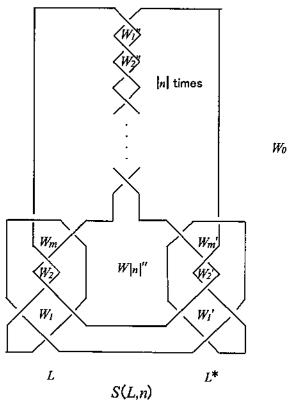

Fig. 2.1

The Goeritz matrix $G(S(L, n))$ associated to $D(S(L, n))$ has the following form:

$$G(S(L,n)) = \begin{pmatrix} G(L) & 0 & 0 & {}^tX \\ 0 & -G(L) & 0 & -{}^tX \\ 0 & 0 & T_{|n|-1} & {}^tY_{|n|-1} \\ X & -X & Y_{|n|-1} & \epsilon \end{pmatrix}.$$

Here, $G(S(L, n))$ is obtained from $\tilde{G}(S(L, n))$ by deleting the row and the column corresponding to the white region W_0, and the last row and the last column of $G(S(L, n))$ correspond to the white region $W''_{|n|}$. Also, $G(L)$ is the Goeritz matrix for L which is an $m \times m$ matrix, ϵ denotes the sign of n and X is a $1 \times m$ matrix. For a positive integer k, Y_k is the $1 \times k$ matrix whose $(1, j)$-th entry is $-\epsilon$ if $j = k$ and is 0 otherwise, and T_k is the $k \times k$ matrix given by

$$T_k = \begin{pmatrix} 2\epsilon & -\epsilon & 0 & 0 & \cdots & 0 \\ -\epsilon & 2\epsilon & -\epsilon & 0 & \cdots & 0 \\ 0 & -\epsilon & 2\epsilon & -\epsilon & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -\epsilon & 2\epsilon & -\epsilon \\ 0 & 0 & \cdots & 0 & -\epsilon & 2\epsilon \end{pmatrix}.$$

First we show the following lemma:

LEMMA 2.1 *For a positive integer k, we have* $\det T_k = (k+1)\,\epsilon^k$ *and* $\sigma(T_k) = k\epsilon$.
Proof. By expanding the first row of T_k, we obtain the following recursion formula:

$$\det T_k = 2\epsilon \det T_{k-1} - \det T_{k-2} \ (k \geq 3),$$

which implies that $\det T_k = (k+1)\,\epsilon^k$. Since

$$\Delta_0 = 1,\ \Delta_1 = \det T_1,\ \Delta_2 = \det T_2,\ \cdots,\ \Delta_k = \det T_k$$

is a σ-series for T_k, we have

$$\sigma(T_k) = \sum_{i=0}^{k-1} \operatorname{sign}(\Delta_i \cdot \Delta_{i+1}) = \sum_{i=0}^{k-1} \epsilon = k\epsilon.$$

(For a σ-series, see [8], [10].)

Q.E.D

Now we prove the following lemma which plays a crucial role in the proof of our theorems:

LEMMA 2.2

(1) *If* $\operatorname{rank}(\ G(L)\quad {}^tX) = \operatorname{rank} G(L) + 1$, *then*
$\operatorname{rank} G(S(L, n)) = 2\operatorname{rank} G(L) + |\,n\,| + 1$ *and* $\sigma(G(S(L, n)) = (|\,n\,| - 1)\,\epsilon$.

(2) *If* $\operatorname{rank}(\ G(L)\quad {}^tX) = \operatorname{rank} G(L)$, *then*
$\operatorname{rank} G(S(L, n)) = 2\operatorname{rank} G(L) + |\,n\,|$ *and* $\sigma(G(S(L, n)) = |\,n\,|\,\epsilon = n$.

Proof. First we consider the case that $\operatorname{rank}(G(L)\quad {}^tX) = \operatorname{rank} G(L) + 1$. Since $G(L)$ is a symmetric matrix, there exists an orthogonal matrix P such that

$$ {}^tPG(L)P = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_2 & & & & 0 & \\ & & \ddots & & & & \\ & & & \lambda_r & & & \\ & & & & \lambda_{r+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_m \end{pmatrix},$$

where $\lambda_1,\lambda_2,\cdots,\lambda_r$ are non-zero real numbers and $\lambda_{r+1},\cdots,\lambda_m$ are 0. Note that $r =$ rank$G(L)$.
Let

$$Q = \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & E_{|n|-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then

$${}^tQA(S(L,n))Q = \begin{pmatrix} {}^tPG(L)P & 0 & 0 & {}^t(XP) \\ 0 & -{}^tPG(L)P & 0 & {}^t(-XP) \\ 0 & 0 & T_{|n|-1} & {}^tY_{|n|-1} \\ XP & -XP & Y_{|n|-1} & \epsilon \end{pmatrix}.$$

Since

$${}^tP(\ G(L)\quad {}^tX\ )\begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix} = (\ {}^tPG(L)P\quad {}^t(XP)\ ),$$

it follows that

$$\mathrm{rank}(\ {}^tPG(\mathrm{L})P\quad {}^t(XP)\ ) = \mathrm{rank}(G(L)\quad {}^tX) = \mathrm{rank}G(L) + 1 = \mathrm{rank}({}^tPG(L)P){+}1.$$

If we put $XP = (x_1,x_2,\cdots,x_r,x_{r+1},\cdots,x_{\mathrm{m}})$, then there exists an integer r_0 $(r+1 \leq r_0 \leq m)$ such that x_{r_0} is not zero. Hence, there exists a non-singular matrix R such that

$${}^tRG(S(L,n))R = \begin{pmatrix} A_r & 0 & 0 & 0 & 0 \\ 0 & -A_r & 0 & 0 & 0 \\ 0 & 0 & T_{|n|-1} & 0 & 0 \\ 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & 0 & C \end{pmatrix},$$

where

$$A_r = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_r \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and C is the $(2m-2r-1)\times(2m-2r-1)$ zero matrix. Therefore we have

$$\mathrm{rank}G(S(L,n))=\mathrm{rank}({}^tRG(S(L,n))R)=2\mathrm{rank}G(L)+|n|+1.$$

Moreover, by using Lemma 2.1, we obtain

$$\sigma(G(S(L,n)))=\sigma({}^tRG(S(L,n))R)=r-r+\sigma(T_{|n|-1})+\sigma(B)=(|n|-1)\,\epsilon\,.$$

Next we consider the second case. Since $\mathrm{rank}(G(L)\quad {}^tX)=\mathrm{rank}G(L)$, it follows from Theorem 7 in Chapter II of [7] that there exists a $1\times m$ matrix Z such that $G(L){}^tZ={}^tX$. If we put

$$U=\begin{pmatrix} E_m & 0 & 0 & -{}^tZ \\ 0 & E_m & 0 & -{}^tZ \\ 0 & 0 & E_{|n|-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then

$${}^tUG(S(L,n))U=\begin{pmatrix} G(L) & 0 & 0 & 0 \\ 0 & -G(L) & 0 & 0 \\ 0 & 0 & T_{|n|-1} & {}^tY_{|n|-1} \\ 0 & 0 & Y_{|n|-1} & \epsilon \end{pmatrix}.$$

Let $V=\begin{pmatrix} T_{|n|-1} & {}^tY_{|n|-1} \\ Y_{|n|-1} & \epsilon \end{pmatrix}$. Then by using Lemma 2.1 we obtain

$$\det V=\epsilon\det T_{|n|-1}+\epsilon\,(-\epsilon)\det T_{|n|-2}=\epsilon^{|n|}=\epsilon^{n}.$$

This implies that

$$\mathrm{rank}G(S(L,n))=2\,\mathrm{rank}G(L)+|n|\,.$$

Since

$$\Delta_0=1,\ \Delta_1=\det T_1,\ \Delta_2=T_2,\ \cdots,\ \Delta_{|n|-1}=\det T_{|n|-1},\Delta_{|n|}=\det V$$

is a σ-series for V, we have

$$\sigma(V)=\sum_{i=0}^{|n|}\text{sign}\,(\Delta_i\cdot\Delta_{i+1})=\sum_{i=0}^{|n|}\epsilon=|\,n\,|\,\epsilon=n.$$

Hence we obtain

$$\sigma\,(G(S(L,n)))=\sigma(G(L))-\sigma(G(L))+\sigma(V)=n.$$

Q.E.D

§ 3. The Proof for Theorem 1.1 and Theorem 1.2.

In this section, we will prove Theorem 1.1 and Theorem 1.2. Since n is even, for an oriented link L, $S(L, n)$ will be shown in Fig. 3.1.

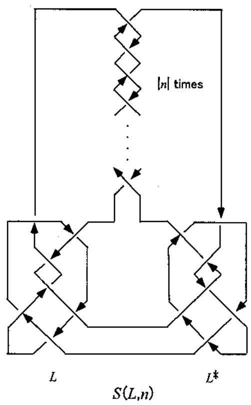

Fig. 3.1

It is easy to show that $\mu\ (D(S(L, n))) = n$.

First of all, we will prove Theorem 1.1. If $\chi\ (S(L, n))\ \equiv\ 0\ (\text{mod}\ 2)$, then

$$\text{rank}G(S(L, n))\ \equiv\ \mu\ (D\ (S(L, n))) = n\ \equiv\ 0\ (\text{mod}\ 2).$$

Lemma 2.2 implies that $\text{rank}(G(L)\quad {}^tX) = \text{rank}G(L)$. Thus we have

$$\sigma\ (G(S(L, n))) =|\ n\ |\ \epsilon = n\quad \text{and}\ \sigma\ (S(L, n)) = n - n = 0.$$

If $\chi\ (S(L, n))\ \equiv\ 1\ (\text{mod}\ 2)$, then

$$\text{rank}G(S(L, n))\ \equiv\ \mu\ (D\ (S(L, n)))+1 = n+1\ \equiv\ 1\ (\text{mod}\ 2).$$

Again, by Lemma 2.2, we have rank($G(L)$ $\ {}^{t}X$) = rank$G(L)$ + 1, which yields

$$\sigma\,(G(S(L,n))) =| n \,|\, \epsilon - \epsilon \quad \text{and} \quad \sigma\,(S(L,n)) = n - \epsilon - n = -\epsilon.$$

This completes the proof for Theorem 1.1.

Finally, we will prove Theorem 1.2. Since det $L =|\det G(L)\,| \neq 0$, rank ($G(L)$ $\ {}^{t}X$) = rank($G(L)$). Hence, Lemma 2.2 implies that

$$\sigma\,(S(L,n)) = \sigma\,(S(L,n)) - \mu\,(D(S(L,n))) = n - n = 0,$$

which completes the proof for Theorem 1.2.

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